M.

QNo→ Define 'dual space' of a normed linear space and Prove that $(l_1)^* = l^\infty$.

Ans→ Dual Space or Conjugate Space :– Let E of normed linear space, we know that, the set B(E,K) of all Continuous linear functionals on E is it self a linear space if be define the norm of a Continuous linear function T by setting,

$$\|T\| = \sup_{\substack{x \in E \\ x \neq 0}} \frac{|T(x)|}{\|x\|}, \quad T \in B(E,K)$$

It can be easily verified that B(E, K) is normed linear space. It can be easily verified that the set B(E, K) is Banach space; B(E, K) is denoted by E^* & is known as Conjugate or dual space of E.

In other words, defn :– Let E be a normed linear space over a field K. Then the set B(E, K) of all Continuous linear functionals on E is a Banach space with respect to Pointwise linear operations and the norm defined by

$$\|T\| = \sup_{\substack{x \in E \\ x \neq 0}} \frac{|T(x)|}{\|x\|}, \quad T \in B(E,K).$$

The set B(E, K) is denoted by E^* and is called the (topological) dual space of the Conjugate space or the adjoint space of E.

Let $f \in (l_1)^*$ (the dual space of l_1^*). It is sufficient to show that there exists $a = (a_n) \in l_1^\infty$ such

Such that,

$$f(x) = \sum_{n=1}^{\infty} x_n a_n \text{ for all } x = (x_n) \in (l_1)^* \text{ and}$$

$$\|f\| = \|a\|$$

$$\text{Let } \delta_m^{(n)} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

and let $\delta^{(k)} = (0, 0, \dots, 1, 0, 0, 0 \dots)$ where the number 1 appears as the kth Co-ordinate and all other Co-ordinates are 0.

Put $a_m = f(\delta^{(m)})$. Let $x = (x_m) \in (l_1)^*$ and let $x = (x_m) \in (l_1)^*$ and let

$$y_n = \sum_{k=1}^{n} x_k (k)$$

$$\text{Then } f(y_n) = f\left(\sum_{k=1}^{n} x_k (k)\right)$$

$$= \sum_{k=1}^{n} x_k f(\delta^{(k)}) = \sum_{k=1}^{n} x_k a_k.$$

Now, $x - y_n = (0, 0, \dots, 0, x_{n+1}, x_{n+2} \dots)$ and thus, $\|x - y_n\| = \text{lub}\{|x_k| : k > n\}$.

Hence, $\underset{n \to \infty}{Lt} \|x - y_n\| = \underset{n \to \infty}{Lt} \text{lub}\{|x_k| : k > n\}$

$= \underset{n \to \infty}{Lt} \sup |x_n| = 0$ Since, $(x_n) \in (l_1)^*$

Thus, $\underset{n \to \infty}{Lt} y_n = x$ in $(l_1)^*$. Since f is Continuous

$$f(x) = \underset{n \to \infty}{Lt} f(y_n) = \underset{n \to \infty}{Lt} \sum_{k=1}^{n} x_k a_k = \sum_{n=1}^{\infty} x_n a_n.$$

We next show that $a = (a_n) \in l^\infty$. To this end

let $\gamma_m^{(n)} = \begin{cases} 1 \text{ if } m \leq n \text{ and } a_m \geq 0 \\ -1 \text{ if } m \leq n \text{ and } a_m < 0 \\ 0 \text{ if } m > n. \end{cases}$

Then, $f(\gamma^{(n)}) = \sum_{m=1}^{l} \gamma_m^{(n)} a_m = \sum_{m=1}^{n} |a_m|$.

Now, $\gamma^{(n)} \in (l_1)^*$ and $\|\gamma^{(n)}\| = 1$

Hence, $|f(\gamma^{(n)})| \leq \|f\|$.

Thus, $\sum_{m=1}^{n} |a_m| \leq \|f\|$ for $n = 1, 2, 3,$ —— (1)

Hence, $a = (a_n) \in l^\infty$. Taking the limit in (1), we have

$$\|a\| = \sum_{n=1}^{\infty} |a_n| \leq \|f\| \quad \text{—— (2)}$$

On the other hand, if $x = (x_n) \in (l_1)^*$ and $\|x\| = 1$

then $|f(x)| = |\sum_{n=1}^{\infty} x_n a_n| \leq \sum_{n=1}^{\infty} |x_n||a_n| \leq \sum_{n=1}^{\infty} |a_n| = \|a\|$

Hence, $\|f\| \leq \|a\|$ —— (3)

From (2) & (3), we have

$\|f\| = \|a\|$.

Therefore, the dual space of $(l_1)^*$ may be identified with l^∞. i.e. $(l_1)^* = l^\infty$.

Q.No → Prove that the dual space of C_0 is l_1.

Verification:- Let $f \in C_0^*$ (the dual space of C_0). It is sufficient to show that there exists $a = (a_n) \in l_1$ such that, there exists $a = (a_n) \in l_1$ such that,

$$f(x) = \sum_{n=1}^{\infty} x_n a_n \text{ for all } x = (x_n) \in C_0 \text{ and } \|f\| = \|a\|$$

$$\text{Let } \delta_m^{(n)} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

and let $\delta^{(K)} = (0, 0, \text{----} 0, 1, 0, 0 \text{--}$

where the number 1 appears as the K'th Co-ordinate and all other Co-ordinates are 0.

Put $a_n = f(\delta^{(n)})$. Let $x = (x_n) \in C_0$ and let

$$y_n = \sum_{K=1}^{n} x_K (K).$$

Then, $f(y_n) = f\left(\sum_{K=1}^{n} x_K (K)\right) = \sum_{K=1}^{n} x_K f(\delta^{(K)}) = \sum_{K=1}^{n} x_K a_K.$

Now, $x - y_n = (0, 0, \dots 0, x_{n+1}, x_{n+2}, \dots)$ and thus
$\|x - y_n\| = \text{lub}\{|x_k| : k > n\}$.
Hence, $\underset{n\to\infty}{\text{Lt}} \|x - y_n\| = \underset{n\to\infty}{\text{Lt}} \text{ lub}\{|x_k| : k > n\}$

$= \underset{n\to\infty}{\text{Lt}} \sup |x_n| = 0$ Since, $(x_n) \in C_0$.

Thus, $\underset{n\to\infty}{\text{Lt}} y_n = x$ in C_0. Since f is continuous

$$f(x) = \underset{n\to\infty}{\text{Lt}} f(y_n) = \underset{n\to\infty}{\text{Lt}} \sum_{k=1}^{n} x_k a_k = \sum_{n=1}^{\infty} x_n a_n.$$

We next show that $a = (a_n) \in l_1$, To this end

let $\gamma_m^{(n)} = \begin{cases} 1 & \text{if } m \leq n \text{ \& } a_m \geq 0 \\ -1 & \text{if } m \leq n \text{ \& } a_m < 0 \\ 0 & \text{if } m > n. \end{cases}$

Then, $f(\gamma^{(n)}) = \sum_{m=1}^{\infty} \gamma_m^{(n)} a_m = \sum_{m=1}^{n} |a_m|$.

Now, $\gamma^{(n)} \in C_0$ and $\|\gamma^{(n)}\| = 1$.
Hence, $|f(\gamma^{(n)})| \leq \|f\|$

Thus, $\sum_{m=1}^{n} |a_m| \leq \|f\|$ for $n = 1, 2, 3.$ —— (1)

Hence, $a = (a_n) \in l_1$. Taking the limit in (1), we have

$$\|a\| = \sum_{n=1}^{\infty} |a_n| \leq \|f\| \text{ —— (2)}$$

On the other hand, if $x = (x_n) \in C_0$ & $\|x\| = 1$.
then, $|f(x)| = \left|\sum_{n=1}^{\infty} x_n a_n\right| \leq \sum_{n=1}^{\infty} |x_n| |a_n| \leq \sum_{n=1}^{\infty} |a_n| = \|a\|$

Hence, $\|f\| \leq \|a\|$ —— (3)

From, (2) & (3), we have

$\|f\| = \|a\|$.

Therefore, the dual space of c_0 may be identified with l_1.